# MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2023-2024

## Séance 9 : Algèbres de Lie (III) et représentations de $su(3)$

### 1. $\mathfrak{su}(2)$ irreps inside $\mathfrak{su}(3)$ representations

Let us collect here a couple of useful basic results. For $\mathfrak{su}(3)$, the fundamental weights $\mu^1, \mu^2$ and the simple roots $\alpha_1, \alpha_2$ are

$$\mu^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right) \ , \quad \mu^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right) \ , \quad \alpha_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \ , \quad \alpha_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \ .$$

A representation $(m,n)$ has as highest weight $m\mu^1 + n\mu^2$, and the dimension is $(m+1)(n+1)(m+n+2)/2$. A given representation can be built starting from the highest weight and building chains of weights generated by simple roots (we will assume the basic representations to be known, it is a good exercise to obtain them by this method!). The $\alpha_i$-chain through a weight $\mu$ is

$$\mu - q\alpha_i, \mu - (q-1)\alpha_i, \ldots, \mu - \alpha_i, \mu, \mu + \alpha_i, \ldots, \mu + (p-1)\alpha_i, \mu + p\alpha_i \ ,$$

with

$$-(p-q) = 2\frac{\alpha_i \cdot \mu}{\alpha_i^2} \ .$$

These chains can be built using any root, not only simple ones. We denote by $\beta$ a general root. Then, starting from a vector with weight $\mu$, we move up and down the chain by acting with the corresponding raising and lowering operators of the $\mathfrak{su}(2)_\beta$ algebra associated with the root $\beta$:

$$E^{\pm} = \frac{1}{|\beta|} e_{\pm\beta} \ , \qquad E^3 = \frac{1}{|\beta|^2} \beta \cdot h \ ,$$

where $\beta \cdot h = \sum_i \beta_i h_i$, $\beta_i$ being the $i$-th component of the root $\beta$. We will be interested in this question in the chains generated by the highest weight root of $\mathfrak{su}(3)$, which is $\beta = \alpha_1 + \alpha_2 = (1,0)$. The corresponding $\mathfrak{su}(2)_\beta$ generators are $E^{\pm} = e_{\pm\beta}$ and $E^3 = h_1$.

#### Section a)

The weight diagram of $\mathbf{3} = (1,0)$ is the well known triangle, with weights

$$(1/2, \sqrt{3}/6) \ , \quad (-1/2, \sqrt{3}/6) \ , \quad (0, -\sqrt{3}/3) \ .$$

We provide a plot of these weights below, together with the $\beta$-chains. Starting from the highest weight $\mu^1 = (1/2, \sqrt{3}/6)$, we get for the $\beta$-chain through it

$$-(p-q) = 2\frac{\beta \cdot \mu^1}{\beta^2} = 1 \ .$$

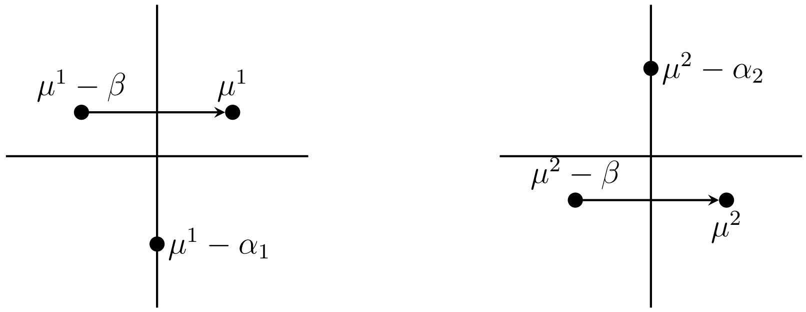



Figure 1: Weights of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations, with the ones forming irreducible representations of $\mathfrak{su}(2)_\beta$ connected by arrows in the direction of $\beta = (1, 0)$.

Furthermore, $p = 0$ because $\mu^1 + \beta$ cannot be a weight ($\mu^1$ is the highest weight of this representation and $\beta$ is a positive root). So $q = 1$, and we see that this chain connects $\mu^1 = (1/2, \sqrt{3}/6)$ and $\mu^1 - \beta = (-1/2, \sqrt{3}/6)$. It is a spin-1/2 chain, and it is nice to check this explicitly by acting in the states $|\mu\rangle$ with $E^3 = h_1$:

$$h_1 \, |1/2, \sqrt{3}/6\rangle = \frac{1}{2} \, |1/2, \sqrt{3}/6\rangle \ , \qquad h_1 \, |-1/2, \sqrt{3}/6\rangle = -\frac{1}{2} \, |-1/2, \sqrt{3}/6\rangle \ .$$

It is clear that the remaining weight, $(0, -\sqrt{3}/3)$, is a singlet for the $\mathfrak{su}(2)_\beta$ algebra (you can compute also $p$ and $q$ here, they are both zero). In Figure 1 you can see the weights and how the associated states form two independent irreducible representations of $\mathfrak{su}(2)_\beta$. It is fairly obvious that things work analogously for the $\bar{\mathbf{3}} = (0, 1)$, so we do not provide further details and instead just include the result in the figure.

The adjoint representation $(1, 1) = \mathbf{8}$ is somewhat trickier. From the previous discussion it is clear that the first component of the weights gives the eigenvalue of $E^3 = h_1$ of the corresponding state, so just looking at the list of weights for the adjoint

$$\begin{array}{cccc} (0,0) \ , & (0,0) \ , & (1/2, \sqrt{3}/2) \ , & (1/2, -\sqrt{3}/2) \ , \\ (1,0) \ , & (-1/2, -\sqrt{3}/2) \ , & (-1/2, \sqrt{3}/2) \ , & (-1, 0) \ , \end{array}$$

we anticipate two spin-1/2 chains (one with second coordinate equal to $\sqrt{3}/2$, the other with $-\sqrt{3}/2$) and a spin-1 chain. This can be confirmed by computing the products between $\beta$ and the corresponding weights and building the chains. The issue arises because we do not know at first what to do with the two $(0, 0)$ weights. The associated vectors span a 2-dimensional space which should be decomposed into two orthogonal directions, one of which should be part of the spin-1 chain $(-1, 0) \to (0, 0) \to (1, 0)$, while the other one should form a singlet. Knowing that we are in the adjoint representation, we can be a bit more explicit. The vectors of this representation are the vectors associated with the roots themselves, $e_\alpha$, plus the two elements of the Cartan subalgebra: $h_1$ and $h_2$. The action is then provided by the commutator. For example, the weight/root $\beta = (1, 0)$ is associated with the vector $e_\beta$, for which we know that

$$[h_1, e_\beta] = \beta(h_1) e_\beta = e_\beta \ , \qquad [h_2, e_\beta] = \beta(h_2) e_\beta = 0 \ .$$

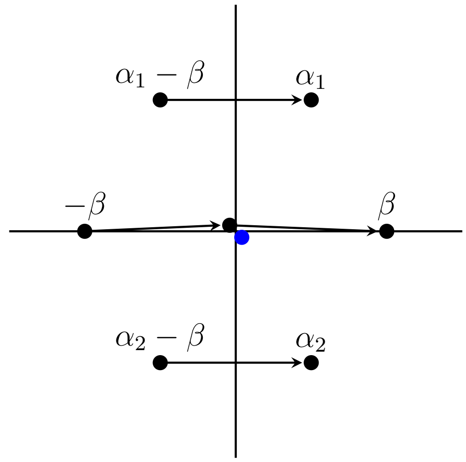



Figure 2: Weights of the adjoint, **8**, with the ones forming irreducible representations of $\mathfrak{su}(2)_\beta$ connected by arrows in the direction of $\beta = (1, 0)$. We indicate the singlet with weights $(0, 0)$ in blue (the associated vector is $h_2$, as shown in the text).

We explicitly see how the eigenvalues $(1, 0)$ appear under the adjoint action of the elements of the Cartan subalgebra. We can go down the $\mathfrak{su}(2)_\beta$ chain similarly by acting with $E^- = e_{-\beta}$:

$$[e_{-\beta}, e_\beta] = -\beta \cdot h = -h_1 \ ,$$

where we used the general result $[e_\alpha, e_{-\alpha}] = \alpha \cdot h$, valid for any root $\alpha$ (upon proper normalization). We see that the element $h_1$ of the Cartan subalgebra is the one contained in the chain $(-1, 0) \to (0, 0) \to (1, 0)$, while $h_2$ is part of a $\mathfrak{su}(2)_\beta$ singlet. This can be further confirmed by the fact that $h_2$ is annihilated by $E^\pm = e_{\pm\beta}$:

$$[e_{-\beta}, h_2] = \beta(h_2) e_{-\beta} = 0 \ , \qquad [e_\beta, h_2] = -\beta(h_2) e_\beta = 0 \ .$$

In Figure 2 you can see the weights of the adjoint representation, with the ones connected by $\mathfrak{su}(2)_\beta$ chains indicated.

## 2. Tensor product representations

We will solve different parts of this question with different techniques: it is a good idea that you try to do all of them with all the techniques to check that you properly understand them. Of course, results must agree.

### Section a)

Let us do a brute force decomposition of $(1, 0) \otimes (0, 1) = \mathbf{3} \otimes \bar{\mathbf{3}}$. From the definition of the tensor product representation of a Lie algebra, it is clear that all weights which are sums of weights of the representations we are tensoring appear. Indeed, if $\mu$ and $\nu$ are weights of the representations $s_1$ and $s_2$,

$$(s_1 \otimes s_2)(h) \, |\mu\rangle \otimes |\nu\rangle = (s_1(h) \, |\mu\rangle) \otimes |\nu\rangle + |\mu\rangle \otimes (s_2(h) \, |\nu\rangle) = (\mu(h) + \nu(h)) \, |\mu\rangle \otimes |\nu\rangle \ .$$

We can then just list all the possible sums of weights of $\mathbf{3}$ and $\bar{\mathbf{3}}$:

$$(0,0),(1,0),\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),(-1,0),(0,0),\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right),(0,0)\ .$$

The highest weight is $(1,0)=\mu^1+\mu^2$ (see question 1 for the form of the fundamental weights). Acting with the operators associated with the simple roots and building the corresponding chains, we would obtain the whole $(1,1)=\mathbf{8}$ irrep (the adjoint). Then, eliminating the corresponding weights from the previous list, we are left with a single $(0,0)$. This weight has an associated vector in the tensor product space which spans a one-dimensional subspace where $\mathfrak{su}(3)$ acts trivially, so this corresponds to the $(0,0)=\mathbf{1}$ representation. We conclude $\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{8}\oplus\mathbf{1}$.

**Section b)**

We will do this by tensor methods. To the $(m,n)$ irrep of $\mathfrak{su}(3)$ we associate tensors with $m$ upper indices and $n$ lower ones, which are totally symmetric in both upper and lower indices and traceless (contraction of any upper index with any lower index gives zero). The tensor product representation is provided by such tensors, and we decompose it by isolating pieces with the desired properties (complete symmetry in upper/lower indices and tracelessness). For this, we can use the $SU(3)$ invariant tensors, $\delta^i_j$ and $\epsilon_{ijk}$. Some useful identities for the Levi-Civita symbol are:

$$\epsilon^{ijk}\epsilon_{lmn}=3!\delta^{[i}_l\delta^j_m\delta^{k]}_n\ ,\quad \epsilon^{ijk}\epsilon_{lmk}=2!\delta^{[i}_l\delta^{j]}_m\ ,\quad \epsilon^{ikm}\epsilon_{jkm}=2!\delta^i_j\ ,\quad \epsilon^{ijk}\epsilon_{ijk}=3!=6\ .$$

Let us decompose $(1,0)\otimes(1,0)=\mathbf{3}\otimes\mathbf{3}$. Take $u^i$ and $v^j$ in the $(1,0)$ representation, isolating the symmetric part of the product we get

$$u^iv^j=u^{(i}v^{j)}+u^{[i}v^{j]}=u^{(i}v^{j)}+\delta^{[i}_l\delta^{j]}_m u^lv^m=u^{(i}v^{j)}+\frac{1}{2}\epsilon^{ijk}\left(\epsilon_{lmk}u^lv^m\right)\ .$$

The first term transforms in the $(2,0)=\mathbf{6}$. In the second one, if we transform all indices, the piece within parentheses transforms in the $(0,1)=\bar{\mathbf{3}}$ (notice that $k$ is the only free index there), while $\epsilon^{ijk}$ is an invariant tensor. Therefore, we conclude $\mathbf{3}\otimes\mathbf{3}=\mathbf{6}\oplus\bar{\mathbf{3}}$.

**Section c)**

Let us do it again by means of tensor methods. Take $u^{ij}=u^{(ij)}$ in the $(2,0)=\mathbf{6}$ and $v^k$ in the $(1,0)=\mathbf{3}$. We form the tensor product $u^{ij}v^k$ and extract the fully symmetric part $u^{(ij}v^{k)}$: this will transform in the $(3,0)=\mathbf{10}$. By subtracting this part which is already under control, define $T^{ijk}\equiv u^{ij}v^k-u^{(ij}v^{k)}$ to be the remaining part of the tensor product. Let us analyze it in detail to find its transformation properties:

$$\begin{aligned}T^{ijk}&=\frac{2}{3}u^{ij}v^k-\frac{1}{3}u^{jk}v^i-\frac{1}{3}u^{ki}v^j=\frac{2}{3}u^{i[j}v^{k]}+\frac{2}{3}u^{j[i}v^{k]}\\&=\frac{1}{3}\epsilon^{jkp}\epsilon_{lmp}u^{il}v^m+\frac{1}{3}\epsilon^{ikp}\epsilon_{lmp}u^{jl}v^m=\frac{2}{3}\left(\delta^{(i}_q\epsilon^{j)kp}\right)\left(\epsilon_{lmp}u^{ql}v^m\right)\ .\end{aligned}$$

We have used here the symmetry of the tensor in the $(2,0) = \mathbf{6}$, $u^{ij} = u^{ji}$, and the same trick of the previous section to write antisymmetrized indices as contractions with two epsilon tensors. All in all, in the last expression, the first parenthesis is just a combination of invariant tensors, while the second one transforms in the $(1,1) = \mathbf{8}$ (there is a free upper index $q$, as well as a lower one $p$). We conclude $\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}$.

**Section d)**

It is tricky to do this final one by tensor methods (it is a good exercise to check that you master them, though!). Let us introduce a final way to compute decompositions of tensor product representations of $\mathfrak{su}(3)$, which has a generalization to any $\mathfrak{su}(N)$. It is known as the method of Young tableaux.[1] We will not justify its validity, but just describe it as an algorithm. First we have to know how to represent irreps as a diagram of boxes. The rules are the following:

1. Rows/columns must be always of non-increasing length.
2. For an $(m,n)$ irrep of $\mathfrak{su}(3)$, we have $m$ columns of length 1 and $n$ columns of length 2.

Some examples are the following:

$$(1,0) = \mathbf{3} = \begin{array}{|c|}\hline \ \\ \hline\end{array}\,, \quad (0,1) = \bar{\mathbf{3}} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\,, \quad (1,1) = \mathbf{8} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad (3,0) = \mathbf{10} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}\,.$$

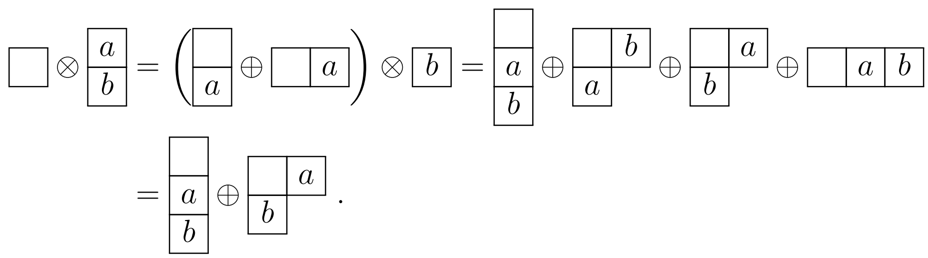
To compute tensor products, we just start from the two original diagrams and follow some set of rules, obtaining as a product the tensor product decomposition:

1. Insert in the diagram of the second factor $a$'s in the first row and $b$'s in the second row.
2. Take, one by one, boxes labelled by $a$, and append them in all allowed ways to the first diagram (the different ways of doing it are separated by $\oplus$). In doing this, there cannot be two or more $a$'s in the same column. Also, if in this process we obtain repeated diagrams, we keep only one copy.
3. Do the same for $b$'s.
4. Starting from the top right box of the resulting diagrams, and moving to the left (and going to the rightmost box of the second row when the first one is completed), do the following count. Compute the total number of $a$'s and $b$'s encountered at any point in the diagram. If the number of $b$'s is larger than the number of $a$'s at some point in the counting, discard the diagram. Otherwise, keep it.

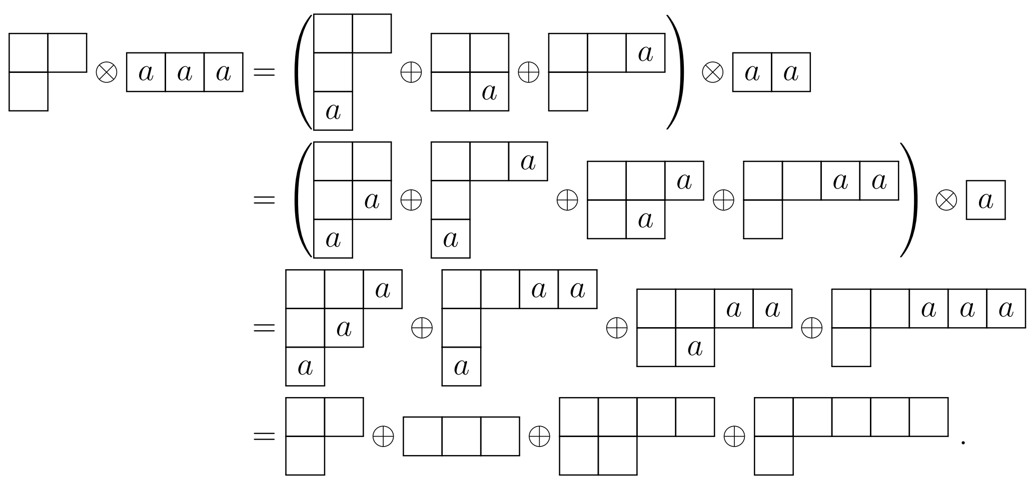
[1] This is not presented in the lecture notes of the course, and as such you should, above all, master the other methods. However, I think it is at least instructive to know that it exists. More information can be found in a compact way here. For more in-depth explanations, consult the standard literature on group theory for physicists.

5. In the final result, columns with three boxes are to be discarded (a diagram containing only three boxes represents the trivial representation $(0,0) = \mathbf{1}$).

This is probably more easily understood with an example. Let us first do $\mathbf{3} \otimes \bar{\mathbf{3}}$, since we already know the result is $\mathbf{8} \oplus \mathbf{1}$. Step by step



Notice the last step, in which we discard the second and fourth diagrams. This is because they do not satisfy the counting described in step 4: when we start from the top right box and keep track of the number of $b$'s and $a$'s we encounter when moving towards the left, we find at some point more $b$'s than $a$'s (in these cases, we already encounter a $b$ before encountering any $a$ in the first step). Letters are superfluous in the final expression and we can just remove them, they were there just for the purpose of the counting. The final two diagrams represent then $\mathbf{1} \oplus \mathbf{8}$, which is the correct result we already knew. We can now try to compute the more difficult case of $\mathbf{8} \otimes \mathbf{10} = (1,1) \otimes (3,0)$ (recall that we do not keep repeated diagrams, so when they appear I already write them only once):



The counting of step 4 is not needed here because there are only $a$'s, which simplifies things a bit. In the last step, we have already discarded columns of length three and deleted the letters, which are not important any more. From this final expression, we read $(1,1) \otimes (3,0) = (1,1) \oplus (3,0) \oplus (2,2) \oplus (4,1)$. We can write it in terms of dimensions also, $\mathbf{8} \otimes \mathbf{10} = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{27} \oplus \mathbf{35}$.

**3. Tensor methods to build irreducible representations**

**Section a)**

We act with $h_1$ and $h_2$ following the rule given in the question (corresponding to how the algebra $\mathfrak{su}(3)$ acts on the tensors corresponding to the $(m,n)$ irrep). Since these are diagonal operators, the expression simplifies to

$$\begin{aligned}\delta_{h_1} v^{1\ldots 3}_{1\ldots 3} &= \left[m_1(h_1)^1{}_1 + m_2(h_1)^2{}_2 + m_3(h_1)^3{}_3 - n_1(h_1)^1{}_1 - n_2(h_1)^2{}_2 - n_3(h_1)^3{}_3\right] v^{1\ldots 3}_{1\ldots 3} \\ &= \frac{1}{2}\left(m_1 - n_1 - m_2 + n_2\right) v^{1\ldots 3}_{1\ldots 3} \ , \\ \delta_{h_2} v^{1\ldots 3}_{1\ldots 3} &= \left[m_1(h_2)^1{}_1 + m_2(h_2)^2{}_2 + m_3(h_2)^3{}_3 - n_1(h_2)^1{}_1 - n_2(h_2)^2{}_2 - n_3(h_2)^3{}_3\right] v^{1\ldots 3}_{1\ldots 3} \\ &= \frac{\sqrt{3}}{6}\left(m_1 - n_1 + m_2 - n_2 - 2m_3 + 2n_3\right) v^{1\ldots 3}_{1\ldots 3} \ .\end{aligned}$$

Notice that we did not write all the indices explicitly in $v^{1\ldots 3}_{1\ldots 3}$. As explained in the question, there are $m_1$ 1's, $m_2$ 2's and $m_3$ 3's in upper position; as well as $n_1$ 1's, $n_2$ 2's and $n_3$ 3's in lower position ($m_1+m_2+m_3 = m$, $n_1+n_2+n_3 = n$). We thus obtain that the weight of this vector is

$$\nu = (\nu^1, \nu^2) = \left(\frac{1}{2}\left(m_1 - n_1 - m_2 + n_2\right), \frac{\sqrt{3}}{6}\left(m_1 - n_1 + m_2 - n_2 - 2m_3 + 2n_3\right)\right) \ .$$

**Section b)**

The weight must be part of the lattice of weights, spanned by the fundamental weights given in the question $\mu^1$ and $\mu^2$. Thus, it must be of the form $\nu = a_1\mu^1 + a_2\mu^2$. Equation with the expression of the previous section, one obtains the values of $a_1$ and $a_2$,

$$\nu = \left(m_1 - n_1 - m_3 + n_3\right)\mu^1 + \left(n_2 - m_2 + m_3 - n_3\right)\mu^2 \ .$$

**Section c)**

Since the representation is the one with Dynkin indices $(m,n)$, the highest weight is $\mu_{HW} = m\mu^1 + n\mu^2 = (m_1+m_2+m_3)\mu^1 + (n_1+n_2+n_3)\mu^2$. All the remaining weights must be obtained by subtracting the simple roots as $\mu_{HW} - b_1\alpha_1 - b_2\alpha_2$, where $\alpha_1 = 2\mu^1 - \mu^2$ and $\alpha_2 = 2\mu^2 - \mu^1$. Equating $\nu$ as in the previous section to this expression and solving for $b_1$ and $b_2$, we obtain

$$\nu = \mu_{HW} - \left(n_1 + m_2 + m_3\right)\alpha_1 - \left(n_1 + m_2 + n_3\right)\alpha_2 \ .$$

**3. Tensor methods to build irreducible representations II**

This builds on the previous question. Notice that the highest weight is obtained for $n_1 = m_2 = m_3 = n_3 = 0$. It must then be $m = m_1$ and $n = n_2$, so the component associated with the highest weight is $v^{1\ldots 1}_{2\ldots 2}$ (all upper components equal to 1, all lower components equal to 2). For our particular example of the $(2,1)$ representation, the component associated with the highest weight is $v^{11}_{2}$. Notice now the following set of rules, which arise from the last expression of the previous question:

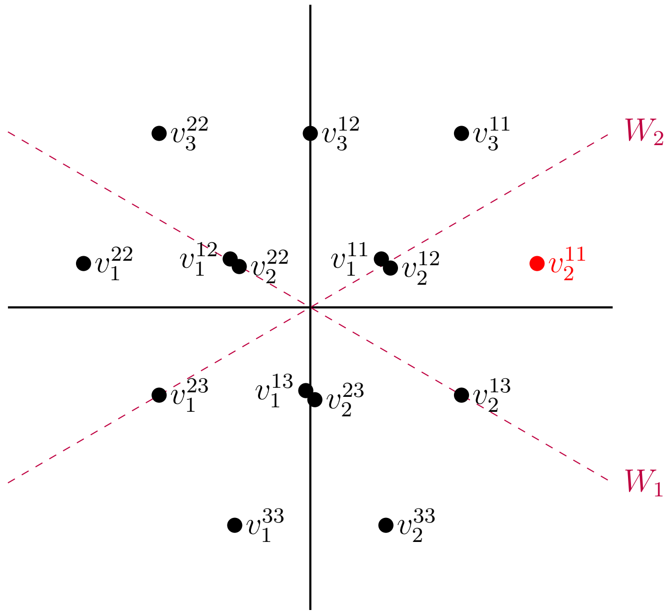



Figure 3: Weights of the representation $(2,1) = \mathbf{15}$, obtained by tensor methods as described in the main text. In this case, we write associated to each weight the corresponding *state* transforming with that weight. This means that we write the only non-zero component of the tensor transforming with the weight (tensors are fully symmetric and traceless, so components related to the written one by this constraints are also non-zero). In red, the highest weight of the representation.

- Changing $1 \to 2$ in upper components decreases the weight by $\alpha_1 + \alpha_2 = (1, 0)$.
- Changing $1 \to 3$ in upper components decreases the weight by $\alpha_1 = (1/2, \sqrt{3}/2)$.
- Changing $2 \to 1$ in lower components decreases the weight by $\alpha_1 + \alpha_2 = (1, 0)$.
- Changing $2 \to 3$ in lower components decreases the weight by $\alpha_2 = (1/2, -\sqrt{3}/2)$.

This is almost enough to build the weight diagram. The only extra thing we have to take into account is that the tensors are traceless, so $v_k^{ik} = 0$ (we are summing over $k$). In other words, not all components of the tensor are independent, because $v_3^{i3} = -v_1^{i1} - v_2^{i2}$ for all $i = 1, 2, 3$. Therefore, when applying the previous rules, if we obtain a state of the form $v_3^{i3}$ we will not consider it as an independent one. Other than that, just applying the previous rules produces the weight diagram in Figure 4. Notice how we indeed get a 15-dimensional representation, as we know we should by applying the general formula. The highest weight is clearly dominant, since $(2\mu^1 + \mu^2) \cdot \alpha_1 = 1 \geq 0$ and $(2\mu^1 + \mu^2) \cdot \alpha_2 = 1/2 \geq 0$. Finally, the Weyl group is generated by reflections perpendicular to the roots, thus transforming points $\nu$ as $\nu \to \nu - 2\frac{\mu \cdot \alpha}{\alpha^2}\alpha$. This group can be shown to be generated by only considering the reflections associated to the simple roots, so we draw the Weyl axes associated to the simple roots $W_1$ and $W_2$, and note that the diagram is indeed invariant under reflections along them.

**4. Tensor methods to build irreducible representations III**

This question is totally analogous to the previous one. We work now with the $(4,1) = \mathbf{35}$ representation, so tensors $v_m^{ijkl}$ provide the representation space. Recall that they are totally symmetric and traceless, so $v_3^{3ijk} = -v_1^{1ijk} - v_2^{2ijk}$ as before. The weight diagram is obtained by applying the rules presented in the previous question starting from the highest weight state: that with only $v_2^{1111}$ non-zero. The weight is $\mu_{HW} = 4\mu^1 + \mu^2 = (5/2, \sqrt{3}/2)$. We get the following diagram:

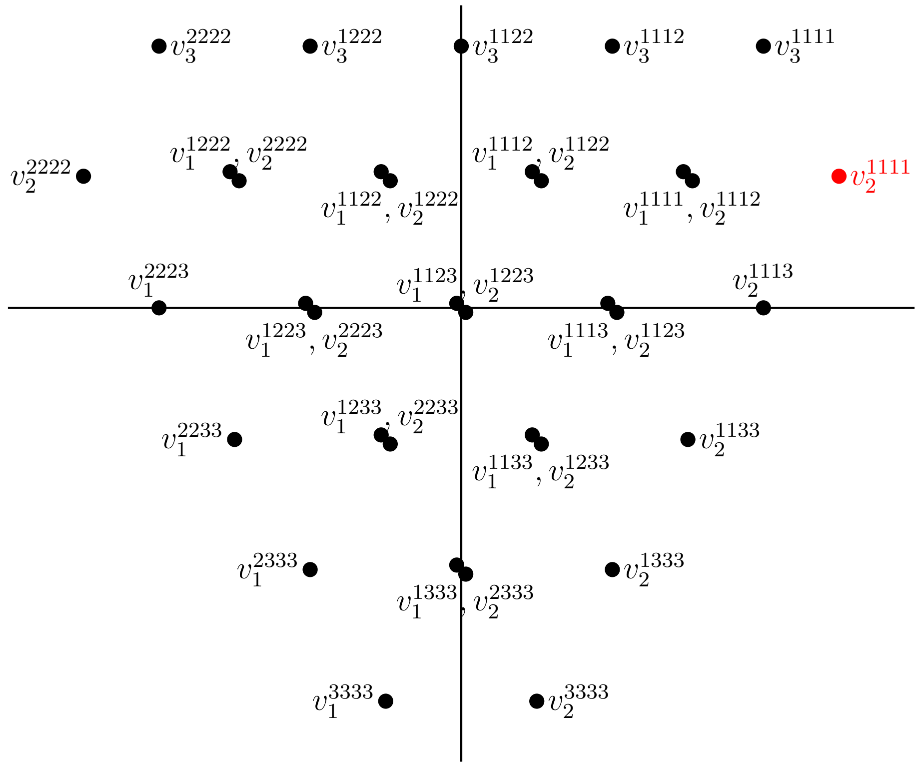



Figure 4: Weights of the representation $(4,1) = \mathbf{35}$, obtained by tensor methods as described in the previous question text. Again, we write associated to each weight the corresponding *state* transforming with that weight. This means that we write the only non-zero component of the tensor transforming with the weight (tensors are fully symmetric and traceless, so components related to the written one by this constraints are also non-zero). In red, the highest weight of the representation.